Carving out a Proof Theory from Cedille’s Core

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What might have Cedille looked like in another universe?
Motivating Examples

How do we build subsets like the set of Even natural numbers in a conceptually intuitive way?

How do we construct inductive types without efficiency problems with lambda encoded data?
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How do we construct inductive types without efficiency problems with lambda encoded data?
Refresher on Dependent Types

$f : \mathbb{N} \rightarrow \mathbb{N}$

function between natural numbers

$P : \mathbb{N} \rightarrow \text{Set}$

predicate on natural numbers

$g : (A : \text{Set}) \rightarrow \text{List } A$

(parametric) function from types to lists

$h : (n : \mathbb{N}) \rightarrow P n$

(dependent) function from naturals to instantiated predicate
$f : \mathbb{N} \rightarrow \mathbb{N}$

given function between natural numbers

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Constructing the Set of Even Numbers

\[ \text{IsEven} : \mathbb{N} \rightarrow \text{Set} \]

any sound way of representing evenness of a natural

\[(a : A) \times P \ a\]

is how we will write \textit{dependent pairs}

\[ \text{Even} = (n : \mathbb{N}) \times \text{IsEven } n \]

pair a natural with evidence it’s even
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How to connect Even and Nat?

\[ \iota : \text{Even} \to \mathbb{N} \]

ask a mathematician, and they say there should be an \textit{injection} from Even to \( \mathbb{N} \)

\[ \iota \equiv \text{fst} \quad 2 : \mathbb{N} \neq 2 : \text{Even} \]

with the current definition

\[ \iota \equiv \text{id} \]

can we change things so that the injection is equal to the identity?
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Quantification w.r.t. what?

$h : (n : \mathbb{N}) \rightarrow P \ n$

via the Curry-Howard correspondence, this is a universal quantification, but what is the domain? The natural numbers? No. The domain is the *proofs* for the given type.

For Even, it is the *proofs* that the element is an even natural, and hence a pair.
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Quantifying over Something New

| · | : Proofs → Objects

*erasure* constructs an *object* (or an *individual*) from a proof

Change conversion from $t =_{\beta} s$ to

$$\exists t' \ s'. \ t \rightarrow_{\beta} t' \land s \rightarrow_{\beta} s' \land |t'| = |s'|$$

Dependent types now quantify over a domain of *objects* instead of proofs.
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Modifying Dependent Pairs

\[
\mathbb{N} \times \text{IsEven } n 
\xrightarrow{\text{pair } n n_e} |n| = |n_e|
\]

\[
|\text{pair } n n_e| = |n|
\]
Modifying Even

\{ x \in \mathbb{N} \mid \exists k, \; 2k = x \}\)

\((x : \mathbb{N}) \times \text{IsEven } x\)

\((x : \mathbb{N}) \cap \text{IsEven } x\)
Is it too hard to inhabit?

Is it too difficult to find types that can be intersected to meet the side-condition?

\[ \Lambda x. t : (x : A) \Rightarrow P x \quad |\Lambda x. t| = |t| \quad x \notin FV(|t|) \]

Add erased functions (or object-irrelevant functions)

Let’s us mark indices as erased, so that the objects of Vec A n are the same objects as List A, additionally all types become erased
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To derive inductive types, we also need equality
Take the standard Martin-Löf Identity Type

\[ J - A - P - x - y - r - w \]

Types must be erased, but also mark indices as erased (critically, the equality evidence \( r \) cannot be erased)

\[ \text{refl} - x \]

Opportunity: we can mark the input to \( \text{refl} \) erased, there can only be one object corresponding to it!
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\[ J \cdot A \cdot P \cdot x \cdot y \cdot r \cdot w \]

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Need to add a rule to reason about dependent intersections

\[
\text{promote: } (x \ y : (a : A) \cap P a) \Rightarrow x.1 =_A y.1 \rightarrow x =_{(a:A)\cap P a} y
\]

Compensates for lack of an induction rule
The system as described is strong enough to derive inductive types (via an impredicative church encoding)

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Inductive Types are Derivable

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Looking for a postdoc for Fall 2024

Q & A